MECHANICAL GENERATION OF CERENKOV RADIATION IN A CONTACTING ELASTIC CONDUCTOR AND A FLUID, IN AN ISOTROPIC MAGNETIC FIELD

(MEKHANICHESKAIA GENERATSIIA CHERENKOVSKOGO IZLUCHENIIA V KONTAKTIRUIUSHCHIKH UPRUGOM PROVODNIKE I ZHIDKOSTI, V IZOTROPNOM MAGNITNOM POLE)

PMM Vol.28, № 5, 1964, pp. 862-867

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(Received July 12, 1963)

The two-dimensional problem of generating Cerenkov radiation in an ideally conducting fluid and an ideally conducting elastic conductor in a magnetic field by means of a moving mechanical impulse travelling in the contact plane of media is examined. The case of isotropic action of primary magnetic field is studied. Solutions are given for the two-dimensional case by the method of characteristics without accounting for dispersion.

In the first section initial equations and boundary conditions are developed. In the second section the case of supermagnetosonic velocity in both media, in the third section the case of intermagnetosonic velocity in the elastic conductor and supermagnetosonic velocity in the fluid, and in the fourth section the case of supermagnetosonic velocity in the fluid and submagnetosonic velocity in the elastic conductor (it is assumed that the velocity of transverse waves in the elastic body is greater than the magnetosonic velocity in the fluid) are examined.

1. An ideal elastic conductor occupying the lower half-space (Fig.1) and an ideally conducting compressible ideal liquid L occupying the upper half-space will be examined. The normal force P acts on the surface of the elastic conductor. This normal force travels with the velocity v_0 in the direction of the x_1-axis. The primary field acts isotropically in the direction of the x_2-axis. We denote by $|\mathbf{H}| = H_2 = H$. The system of reference is shown in Fig.1. The gravitational field in the fluid is neglected. We will assume that the fluid is initially under tension to such an extent that it can sustain negative pressures. In addition, for the sake of definiteness, we assume that

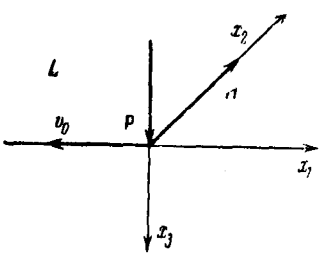

Fig. 1

$$a_1^* > a_2 > a_0^* \tag{1.1}$$
$$a_1^{*2} = a_1^2 + \varkappa^2, \qquad a_0^{*2} = a_0^2 + \varkappa_0^2$$
$$a_1 = \left(\frac{\lambda + 2G}{\rho}\right)^{1/2}, \qquad a_2 = \left(\frac{G}{\rho}\right)^{1/2}$$
$$\varkappa^2 = \frac{\mu H^2}{4\pi\rho}, \qquad \varkappa_0^2 = \frac{\mu_0 H^2}{4\pi\rho_0}$$

Here a_0 is the velocity of the acoustic wave in the fluid, λ and G are elastic constants of the elastic conductor, ρ and ρ_0 are densities of the elastic conductor and the fluid respectively, and μ and μ_0 are magnetic permeabilities of the elastic conductor and the fluid.

Assumption (1.1) is not essential, solution for other inequalities is analogous. For simplification we will set $\mu \approx \mu_0 \approx 1$ in the following material. We will also take advantage of nonrelativistic equations for magnetohydrodynamics [1] and for an elastic conductor $(v_0^2 / c^2 \ll 1)$. According to [2] linearized equations of magnetoelasticity for an ideal elastic conductor and a fluid will have the form

$$a_2^2\nabla^2\mathbf{u} + (a_1^2 - a_2^2)\,\text{grad div}\,\mathbf{u} + \frac{\mu}{4\pi\rho}[\text{rot rot}\,(\mathbf{u}\times\mathbf{H})]\times\mathbf{H} - \frac{\partial^2\mathbf{u}}{\partial t^{\circ 2}} = 0 \tag{1.2}$$

where

$$\mathbf{E} = -\frac{\mu}{c}\left(\frac{\partial\mathbf{u}}{\partial t^\circ}\times\mathbf{H}\right), \qquad \mathbf{h} = \text{rot}\,(\mathbf{u}\times\mathbf{H}) \tag{1.3}$$

$$a_0^2\,\text{grad div}\,\mathbf{u}' + \frac{1}{4\pi\rho_0}[\text{rot rot}\,(\mathbf{u}'\times\mathbf{H})]\times\mathbf{H} - \frac{\partial^2\mathbf{u}'}{\partial t^{\circ 2}} = 0 \tag{1.4}$$

where

$$\mathbf{E}' = -\frac{\mu_0}{c}\left(\frac{\partial\mathbf{u}'}{\partial t^\circ}\times\mathbf{H}\right); \qquad \mathbf{h}' = \text{rot}\,(\mathbf{u}'\times\mathbf{H}) \tag{1.5}$$

Equations (1.2) and (1.4) were simplified with respect to $\partial/\partial t^\circ$ under the assumption that $\partial\mathbf{u}/\partial t^\circ \neq 0$ and $\partial\mathbf{u}'/\partial t^\circ \neq 0$. In the case where $|\mathbf{H}| = H_2 = H$ equations (1.2) and (1.4) acquire the form

$$a_2^2\nabla^2\mathbf{u} + (a_1^{*2} - a_2^2)\,\text{grad div}\,\mathbf{u} - \frac{\partial^2\mathbf{u}}{\partial t^{\circ 2}} = 0, \quad a_0^{*2}\,\text{grad div}\,\mathbf{u}' - \frac{\partial^2\mathbf{u}'}{\partial t^{\circ 2}} = 0 \tag{1.6}$$

Boundary conditions on the surface of the half-space will have the form

$$\sigma_{3^\circ 3^\circ} + T_{3^\circ 3^\circ} - \sigma'_{3^\circ 3^\circ} - T'_{3^\circ 3^\circ} = -P\delta\,(x_1^\circ + v_0 t^\circ), \qquad \sigma_{3^\circ 1^\circ} = 0, \qquad u_{3^\circ} = u_{3^\circ}' \tag{1.7}$$

Here $\sigma_{i^\circ k^\circ}$ is the mechanical stress in the elastic conducter, $\sigma'_{i^\circ i^\circ}$ is the pressure in the fluid, $T_{i^\circ k^\circ}$ and $T'_{i^\circ k^\circ}$ are Maxwell's stress tensors in the solid body and in the fluid.

Maxwell's tensors are expressed in the following formulas for $\mu = \mu_0 \approx 1$

$$T_{i^\circ k^\circ} = \frac{1}{4\pi}[H_{k^\circ}h_{i^\circ} + H_{i^\circ}h_{k^\circ} - \mathbf{H}\mathbf{h}\delta_{i^\circ k^\circ}]$$
$$T'_{i^\circ k^\circ} = \frac{1}{4\pi}[H_{k^\circ}h_{i^\circ}' + H_{i^\circ}h_{k^\circ}' - \mathbf{H}\mathbf{h}'\delta_{i^\circ k^\circ}] \tag{1.8}$$

In the case where $|\mathbf{H}| = H_2 = H$, and after expressing $\mathbf{h}$ and $\mathbf{h}'$ through through $\mathbf{u}$ and $\mathbf{u}'$, conditions (1.7) take finally the form

$$\rho\,(a_1^{*2} - 2a_2^2)\,u_{1^\circ,1^\circ} + \rho a_1^{*2} u_{3^\circ,3^\circ} - \rho_0 a_0^{*2} u_{1^\circ,1^\circ} - \rho_0 a_0^{*2} u_{3^\circ,3^\circ} = = -P\delta\,(x_1^\circ + v_0 t^\circ)$$

$$u_{1^\circ,3^\circ} + u_{3^\circ,1^\circ} = 0, \qquad u_{3^\circ} - u_{3^\circ}' = 0 \tag{1.9}$$

It is assumed here that $u_{2^\circ} = u_{2^\circ}' = 0$ since in the case of the two-dimensional problem stresses are independent of x_2.

Further, let us introduce, analogously to [3], the followng potentials:

$$u_{1^\circ} = \Phi_{,1^\circ} + \Psi_{,3^\circ}, \qquad u_{3^\circ} = \Phi_{,3^\circ} - \Psi_{,1^\circ}, \qquad \mathbf{u}' = \operatorname{grad}\varphi \tag{1.10}$$

Then the system of equations (1.6) is reduced to the folloing, separate for each potential, equations

$$\nabla^2\Phi - \frac{1}{a_1^{*2}}\Phi_{,t^\circ t^\circ} = 0, \qquad \nabla^2\Psi - \frac{1}{a_2^2}\Psi_{,t^\circ t^\circ} = 0, \qquad \nabla^2\varphi - \frac{1}{a_0^{*2}}\varphi_{,t^\circ t^\circ} = 0 \tag{1.11}$$

We introduce a new reference system [3]

$$x_1 = x_1^\circ + v_0 t^\circ, \qquad x_3 = x_3^\circ, \qquad t = t^\circ \tag{1.12}$$

We shall examine the established process, then Equations (1.11) and boundary conditions (1.9) will assume the form

$$\Phi_{,33} - \alpha_1^2\Phi_{,11} = 0, \qquad \Psi_{,33} - \alpha_2^2\Psi_{,11} = 0, \qquad \varphi_{,33} - \alpha_0^2\varphi_{,11} = 0 \tag{1.13}$$

$$(\alpha_1^2 = v_0^2 / a_1^{*2} - 1, \; \alpha_2^2 = v_0^2 / a_2^2 - 1, \; \alpha_0^2 = v_0^2 / a_0^{*2} - 1)$$

$$\rho\,(a_1^{*2} - 2a_2^2)\,u_{1,1} + \rho a_1^{*2} u_{3,3} - \rho_0 a_0^{*2} u_{1,1}' - \rho_0 a_0^{*2} u_{3,3}' = -P\delta\,(x_1)$$

$$u_{1,3} + u_{3,1} = 0, \qquad u_3 - u_3' = 0 \tag{1.14}$$

or

$$\rho\,(a_1^{*2} - 2a_2^2)\,\Phi_{,11} + \rho a_1^{*2}\Phi_{,33} - 2\rho a_2^2\Psi_{,13} - \rho_0 a_0^{*2}\varphi_{,11} - - \rho_0 a_0^{*2}\varphi_{,33} = -P\delta\,(x_1) \tag{1.15}$$

$$2\Phi_{,13} + \Psi_{,33} - \Psi_{,11} = 0, \qquad \Phi_{,3} - \Psi_{,1} - \varphi_{,3} = 0$$

When, according to condition (1.1), the inequality $\alpha_0^2 > \alpha_2^2 > \alpha_1^2$ is satisfied, the case of supermagnetosonic velocity in both media occurs for $v_0 > a_1^*$; the case of intermagnetosonic velocity in the elastic conductor and supermagnetosonic velocity in the fluid occurs for $a_2 < v_0 < a_1^*$; the case of submagnetosonic velocity in the elastic conductor and supermagnetosonic velocity in the fluid occurs for $a_0^* < v_0 < a_2$.

These three cases for intervals of velocity change of mechanical impulse v_0 are examined below; in the first of these cases three cones of Cerenkov radiation appear, two in the elastic conductor and one in the fluid, in the second case one cone appears in the elastic conductor and one in the liquid, in the third case one cone appears in the liquid.

2. For supermagnetosonic velocity in both media $\alpha_0, \alpha_1, \alpha_2 > 0$. We will seek solutions of Equations (1.13) in the form

$$\Phi = \Phi(x_1 - \alpha_1 x_3), \qquad \Psi = \Psi(x_1 - \alpha_2 x_3), \qquad \varphi = \varphi(x_1 + \alpha_0 x_3) \quad (2.1)$$

For $x_3 = 0$ we find from the second, the third and the first boundary condition (1.15), respectively,

$$\Phi'' = -\frac{1-\alpha_2^2}{2\alpha_1}\Psi'', \qquad \varphi'' = -\frac{1+\alpha_2^2}{2\alpha_0}\Psi'', \qquad \Psi''(x_1) = \frac{P}{M}\delta(x_1) \quad (2.2)$$

Here primes denote differentiation with respect to the argument (2.3)

$$M = \rho\,[a_1^{*2}(1+\alpha_1^2) - 2a_2^2]\frac{1-\alpha_2^2}{2\alpha_1} - 2\rho a_2^2\alpha_2 - \rho_0 a_0^{*2}(1+\alpha_0^2)\frac{1+\alpha_2^2}{2\alpha_0}$$

Integrating the last equation of (2.2) we find

$$\Psi'(x_1 - \alpha_2 x_3) = \frac{P}{M}H(x_1 - \alpha_2 x_3) \quad (H \text{ is Heaviside's function}) \quad (2.4)$$

The constant of integration is omitted: it can only introduce a constant displacement which will be unessential.

In the following (and also in later Sections) we will operate with displacements not defined exactly in this sense (the derivatives of displacements, apparently, will be defined exactly). After computation of Ψ', $-\Phi'$ and φ' are determined from the other two relationships of (2.1). Then the displacements u_i and u_i' ($i = 1, 3$) are determined with accuracy to arbitrary constants

$$u_1 = -\frac{P}{M}\left[\frac{1-\alpha_2^2}{2\alpha_1}H(x_1 - \alpha_1 x_3) + \alpha_2 H(x_1 - \alpha_2 x_3)\right]$$

$$u_1' = -\frac{P}{M}\frac{1+\alpha_2^2}{2\alpha_0}H(x_1 + \alpha_0 x_3)$$

$$u_3 = \frac{P}{M}\left[\frac{1-\alpha_2^2}{2}H(x_1 - \alpha_1 x_3) - H(x_1 - \alpha_2 x_3)\right] \quad (2.5)$$

$$u_3' = -\frac{P}{M}\frac{1+\alpha_2^2}{2}H(x_1 + \alpha_0 x_3)$$

Components of field vectors in both media are found with the aid of (2.5) from Formulas (1.3) and (1.5) (it is recalled that $\partial(\ldots)/\partial t^\circ$ must be replaced after transformation by $v_0\partial(\ldots)/\partial x_1$). From solutions of (2.5) it is evident that for supermagnetosonic velocity in both media three cones of Cerenkov radiation arise: two in the elastic conductor and one in the fluid (Fig.2). The case is analogous for field components. It follows from solutions of (2.5) that the orders of magnitude of deformations and velocity are $P\delta/M$ and $Pv_0\delta/M$ and, correspondingly, the orders of magnitude of perturbed magnetic and electrical fields in the region of cones are $HP\delta/M$ and $PHv_0\delta/cM$; in

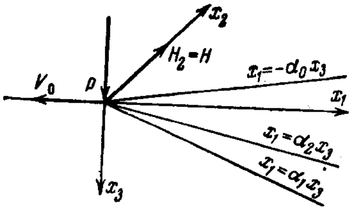

Fig. 2

the determination of the order of magnitude the symbol δ has the significance that it determines the influence function which on multiplication by the general function and after integration yields the solution for a given loading and permits to compare the orders of magnitude of parameters under examination. It follows from the discussion presented that when the loadings result in stresses of the order of the limit of elasticity in the elastic conductor, then the field intensities of Cerenkov radiation have practical orders of magnitude [3 and 4].

3. Let us examine intermagnetosonic velocity of impulse in the elastic conductor and supermagnetosonic velocity in the fluid. For construction of solutions in the case $a_0^* < a_2 < v_0 < a_1^*$ we proceed analogously to the papers [3] and [4], i.e. we utilize solutions of Section 2 introducing instead of $-\alpha_1^2$ the quantity $\beta_1^2 > 0$ or $\alpha_1 = -i\beta_1$. Then the first of Equations (1.13) becomes elliptical.

In connection with this and in accordance with [3] we introduce instead of the function $H(x_1 - \alpha_1 x_3)$ its analytical continuation in solutions (2.5)

$$H(\xi) = H(x_1 + i\beta_1 x_3) = \frac{i}{\pi}\ln \xi + 1 = \frac{i}{\pi}\ln r + 1 - \frac{\varphi}{\pi} \qquad (3.1)$$

where

$$\xi = x_1 + i\beta_1 x_3 = re^{i\varphi} \qquad (3.2)$$

The function $H(\xi)$ satisfies the elliptical equation for Φ and assumes the values of the Heaviside function along the real axis. After determining Ψ and φ from boundary conditions or appropriately selected analytical extensions, we compute the functions $\mathbf{u}$ and $\mathbf{u}'$. Retaining the real parts we obtain the desired solution to the problem. Solutions atisfy the equations and the boundary conditions. After substituting α_1 by $-i\beta_1$ in the expression for M we obtain

$$M = s_1 + is_2 \qquad (3.3)$$

$$s_1 = -\left[2\rho a_2^2\alpha_2 + \rho_0 a_0^{*2}(1+\alpha_0^2)\frac{1+\alpha_2^2}{2\alpha_0}\right], \quad s_2 = \rho[a_1^{*2}(1-\beta_1^2) - 2a_2^2]\,\frac{1-\alpha_2^2}{2\beta_1}$$

Further

$$\frac{1}{M} = n_1 + in_2 \qquad \left(n_1 = \frac{s_1}{s_1^2 + s_2^2}, \quad n_2 = -\frac{s_2}{s_1^2 + s_2^2}\right) \qquad (3.4)$$

From this, after computation of Ψ and φ, according to (1.15), and after substitution into (1.10), or directly from (2.5) on the basis of appropriately selected analytical continuations, and after separation of real parts, we find

$$(3.5)$$

$$u_1 = P\left\{\frac{1-\alpha_2^2}{2\beta_1}\left[\frac{n_1}{\pi}\ln r + n_2\left(1 - \frac{\varphi}{\pi}\right)\right] - n_1\alpha_2 H(x_1 - \alpha_2 x_3) + \frac{\alpha_2 n_2}{\pi}\ln|x_1 - \alpha_2 x_3|\right\}$$

$$u_3 = P\left\{\frac{1-\alpha_2^2}{2}\left[n_1\left(1 - \frac{\varphi}{\pi}\right) - \frac{n_2}{\pi}\ln r\right] - n_1 H(x_1 - \alpha_2 x_3) + \frac{n_2}{\pi}\ln|x_1 - \alpha_2 x_3|\right\}$$

$$u_1' = -P\left[n_1\frac{1+\alpha_2^2}{2\alpha_0}H(x_1 + \alpha_0 x_3) - \frac{n_2}{\pi}\frac{1+\alpha_2^2}{2\alpha_0}\ln|x_1 + \alpha_0 x_3|\right]$$

$$u_3' = -P\left[n_1\frac{1+\alpha_2^2}{2}H(x_1 + \alpha_0 x_3) - \frac{n_2}{\pi}\frac{1+\alpha_2^2}{2}\ln|x_1 + \alpha_0 x_3|\right]$$

It is evident from (3.5) that the solution in the elastic medium and in the fluid consists of a stationary part and a cone of Cerenkov radiation. Stationary disturbances overtake the fronts of disturbances of Cerenkov radiation cones. The orders of magnitude of individual parameters of solutions correspond to those established in the previous section.

The computation of field components in both media by means of relationships (1.3) and (1.5) is not presented here.

4. Let us examine submagnetosonic velocity of impulse in the elastic conductor and supermagnetosonic velocity in the fluid. Just as in Section 3 we introduce the following notation:

$$\alpha_1 = -i\beta_1, \qquad \alpha_2 = -i\beta_2 \tag{4.1}$$

since equations for Φ and Ψ will transform from hyperbolic to elliptical.

Consequently we write in analogy to (4.1)

$$\begin{aligned} H(\xi_1) &= H(x_1 + i\beta_1 x_3) = \frac{i}{\pi}\ln\xi_1 + 1 = \frac{i}{\pi}\ln r_1 + 1 - \frac{\varphi_1}{\pi} \\ H(\xi_2) &= H(x_1 + i\beta_2 x_3) = \frac{i}{\pi}\ln\xi_2 + 1 = \frac{i}{\pi}\ln r_2 + 1 - \frac{\varphi_2}{\pi} \end{aligned} \tag{4.2}$$

where

$$\xi_1 = x_1 + i\beta_1 x_3 = r_1 e^{i\varphi_1}, \qquad \xi_2 = x_1 + i\beta_2 x_3 = r_2 e^{i\varphi_2} \tag{4.3}$$

Substituting (4.1) into (2.5) we find

$$\begin{aligned} M &= s_1 + is_2, \qquad s_1 = -\rho_0 a_0^{*2}(1 + \alpha_0^2)\frac{1 - \beta_2^2}{2\alpha_0} \\ s_2 &= \rho\,[a_1^{*2}(1 - \beta_1^2) - 2a_2^2]\frac{1 + \beta_2^2}{2\beta_1} + 2\rho a_2^2\beta_2 \end{aligned} \tag{4.4}$$

$$\frac{1}{M} = n_1 + in_2, \qquad n_1 = \frac{s_1}{s_1^2 + s_2^2}, \qquad n_2 = -\frac{s_2}{s_1^2 + s_2^2} \tag{4.5}$$

Substitution of these expressions into (2.5) with corresponding analytical continuations (or with functions Ψ and φ determined from boundary conditions) leads to the following result after separation of real parts

$$\begin{aligned} u_1 &= P\left\{\frac{n_1}{\pi}\left[\frac{1 - \beta_2^2}{2\beta_1}\ln r_1 - \beta_2 \ln r_2\right] + n_2\left[\frac{1 + \beta_2^2}{2\beta_1}\left(1 - \frac{\varphi_1}{\pi}\right) - \beta_2\left(1 - \frac{\varphi_2}{\pi}\right)\right]\right\} \\ u_3 &= P\left\{n_1\left[\frac{1 + \beta_2^2}{2}\left(1 - \frac{\varphi_1}{\pi}\right) - \left(1 - \frac{\varphi_2}{\pi}\right)\right] + \frac{n_2}{\pi}\left[\ln r_2 - \frac{1 + \beta_2^2}{2}\ln r_1\right]\right\} \\ u_1' &= -P\left[\frac{1 - \beta_2^2}{2\alpha_0} n_1 H(x_1 + \alpha_0 x_3) - \frac{n_2}{\pi}\frac{1 - \beta_2^2}{2\alpha_0}\ln|x_1 + \alpha_0 x_3|\right] \\ u_3' &= -P\left[\frac{1 - \beta_2^2}{2} n_1 H(x_1 + \alpha_0 x_3) - \frac{n_2}{\pi}\frac{1 - \beta_2^2}{2}\ln|x_1 + \alpha_0 x_3|\right] \end{aligned} \tag{4.6}$$

Solution (4.6) gives only one Cerenkov radiation cone in the fluid (and the stationary part). In the elastic medium the solution has a stationary character (in accordance with the finding that after a sufficiently long time interval the process is established). As before, the derivatives of (4.6) will be determined unambigously; field vectors are computed from (1.3) and (1.5). As far as the orders of magnitude are concerned, remarks in the previous section are applicable.

The case of submagnetosonic velocity in the fluid ($v_0 < a_0^*$) is not examined since it does not yield radiation cones. The stationary solution in this case is also readily obtained on the basis of solution of (2.5).

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Translated by J.R.W.